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Scattering operators on Fock space: III. Euclidean invariant scattering amplitudes and SL(2, ℝ)†

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Abstract. Scattering operators invariant under the simplest spacetime group, namely the two-dimensional Euclidean group E(2), are investigated. Operators that commute with the action of E(2) on a Fock space generated by an infinite-dimensional representation of E(2) are constructed and shown to form a Lie algebra of SL(2, ℝ). Unitary scattering amplitudes which include production reactions are given as matrix elements of the discrete series of representations of SL(2, ℝ).

1. Introduction

The goal of this series of papers is to construct representations of scattering amplitudes that satisfy general quantum mechanical properties such as unitarity, relativistic invariance and crossing. In previous papers only internal symmetry degrees of freedom were considered and the scattering amplitudes constructed had no spacetime dependence. For example, in Klink (1987) SO(N) internal symmetries generate production partial-wave amplitudes related to the matrix elements of SL(2, ℝ). A physical example is SU(2) isospin symmetry, where the three pions, π^+, π^0 and π^-, generate a symmetric Fock space on which the SL(2, ℝ) operators act.

In this paper we want to extend the analysis begun in Klink (1987, hereafter referred to as II) to spacetime groups for which the one-particle space is an infinite-dimensional Hilbert space. Though the obvious group here is the Poincaré group, because of some delicate limits we will work instead with the somewhat simpler two-dimensional Euclidean group E(2) and defer the analysis of the Poincaré group to a succeeding paper. The limits occur when making the transition from Fock spaces generated by finite-dimensional representation spaces of compact internal symmetry groups to Fock spaces generated by infinite-dimensional representation spaces of non-compact spacetime groups.

To study these limits we will start in § 2 with a symmetric Fock space generated by the $(2l+1)$-dimensional representation of SO(3) and in § 3 let l go to infinity, in such a way as to get a representation of E(2). In the limit some SL(2, ℝ) operators are no longer well defined. However, when a transformation to a continuous momentum basis is made, the operators commuting with the E(2) action again form a SL(2, ℝ) Lie algebra. This means that unitary partial-wave amplitudes that are E(2) invariant can again be given as SL(2, ℝ) matrix elements.

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2. SL(2, ℝ) operators dual to the $(2l+1)$-dimensional representation of SO(3)

We start with a $(2l+1)$-dimensional representation space V^l of SO(3). It generates a Fock space $\mathbb{S}(V^l)$ given by

$$\mathbb{S}(V^l)=\sum_{n=0}^{\infty}(V^l\otimes\ldots\otimes V^l)^n_{\mathrm{sym}} \tag{1}$$

which is the direct sum of n-fold symmetric tensor products of V^l. The irreducible representation content L of some of the low-lying particle states are

$$\begin{array}{llllll} & L=0\ldots & l\ldots & 2l\ldots & 3l\ldots & 4l\ldots \\ n=0 & \times & & & & \\ 1 & & \times & & & \\ 2 & \times & \ldots & \times & & \\ 3 & \times & \ldots & & \times & \\ 4 & \times & \ldots & & & \times \\ \vdots & & & & & \end{array} \tag{2}$$

For $n=2$, the two-particle state, $L=0,2,4,\ldots,2l$, so there is no multiplicity. For a general n-particle state, the largest L value is nl, which is also multiplicity free. However, for L values less than nl there will in general be multiplicity, so that, unlike the case of $l=1$ analysed in II, n, L and L_3 do not uniquely fix a state. Notice also the 'anti' thresholds in (2), where, as L gets larger than $2l$, the two-particle states no longer contribute, and for L larger than $3l$ the three-particle states no longer contribute, and so forth. This is the opposite of the Poincaré group, where the two-particle states always contribute, no matter how large the invariant mass is.

To find operators that commute with the action of SO(3) on $\mathbb{S}(V^l)$, it is useful to introduce a holomorphic Hilbert space, HL^2_{2l+1} (Bargmann 1962, Segal 1956), which is isomorphic to $\mathbb{S}(V^l)$. HL^2_{2l+1} consists of holomorphic functions in $(2l+1)$ variables with norm

$$\|f\|^2=\frac{1}{\pi^{2l+1}}\int_{\mathbb{C}^{2l+1}}\mathrm{d}z_l\ldots\mathrm{d}z_{-l}\exp\left(-\sum_{m=-l}^{+l}|z_m|^2\right)|f(z)|^2<\infty. \tag{3}$$

Since HL^2_{2l+1} is isomorphic to $\mathbb{S}(V^l)$, it can also be decomposed into a direct sum of n-particle subspaces. A convenient orthogonal basis for an n-particle subspace is given by $z_{m_1}\ldots z_{m_n}$, where $m_1\ldots m_n$ range over all possible values from $-l$ to $+l$.

Let $D(R)\equiv D^l_{mm'}(R)$ be the matrix element of $R\in\mathrm{SO}(3)$ in V^l. Then the action of SO(3) on elements in HL^2_{2l+1} is given by

$$(\Gamma_R f)(z)=f(D^{-1}(R)z)\qquad f\in HL^2_{2l+1}. \tag{4}$$

The infinitesimal operators are

$$\begin{aligned}(L^l_3f)(z)&=\sum_{m=-l}^{+l}mz_m\frac{\partial}{\partial z_m}f(z)\\ (L^l_\pm f)(z)&=\sum_{m=-l}^{+l}c^\pm_{lm}z_{m\pm1}\frac{\partial}{\partial z_m}f(z)\end{aligned} \tag{5}$$

where $c^\pm_{lm}=[j(j+1)-m(m\pm1)]^{1/2}$.

We now want to find operators that commute with the SO(3) action on HL^2_{2l+1}. Since the matrix elements of SO(3) form a subgroup of SO($2l+1$), a theorem of Howe (1985) says that operators exist that commute with SO($2l+1$) and form a Lie algebra of SL(2, $\mathbb{R}$). The point of this paper is to show that such SL(2, $\mathbb{R}$) operators exist even when l goes to infinity.

For l finite we consider a polynomial $p(z)$ which is invariant under the SO(3) action of equation (5). Such a polynomial can be obtained by extracting the $L=0$ piece of the tensor product of l with itself. Then p is of the form

$$p(z)=\sum_{m=-l}^{+l}(-1)^m z_m z_{-m}. \tag{6}$$

Define

$$(X^l_+ f)(z)=p(z)f(z). \tag{7}$$

Then $[X^l_+, L^l_3]=[X^l_+, L^l_\pm]=0$ because $L^l_3 p=L^l_\pm p=0$. A lowering operator can correspondingly be defined as

$$(X^l_- f)(z)=p(\partial/\partial z)f(z) \tag{8}$$

where $p(\partial/\partial z)$ means $\Sigma(-1)^m(\partial/\partial z_m)(\partial/\partial z_{-m})$. X^l_- also commutes with L^l_3 and $L^l_\pm$, and is the adjoint of X^l_+.

Finally the number operator, given by

$$\hat{n}^l=\Sigma\, z_m\left(\frac{\partial}{\partial z_m}\right) \tag{9}$$

commutes with the SO(3) action of equation (4), has the value n on each n-particle subspace and has commutation relations with $X^l_\pm$ of the form

$$\begin{aligned}&[\hat{n}^l, X^l_\pm]=\pm 2X^l_\pm\\&[X^l_-, X^l_+]=2(2l+1)I+4\hat{n}^l\end{aligned} \tag{10}$$

where I is the identity operator on HL^2_{2l+1}. If X^l_0 is defined as

$$X^l_0=\tfrac{1}{2}(2l+1)+\hat{n}^l \tag{11}$$

the commutation relations become

$$\begin{aligned}&[X^l_0, X^l_\pm]=\pm 2X^l_\pm\\&[X^l_-, X^l_+]=4X^l_0\end{aligned} \tag{12}$$

which are the commutation relations for SL(2, $\mathbb{R}$).

These operators form a reducible representation of the Lie algebra of SL(2, $\mathbb{R}$). To find the connection between irreducible representations of the covering group of SL(2, $\mathbb{R}$) and partial-wave amplitudes of the scattering operator, we transform the SL(2, $\mathbb{R}$) operators $X^l_\pm$, X^l_0 to $\mathbb{S}(V^l)$. Let $\hat{e}_{m_1\ldots m_n}$ be a basis element in the n-particle subspace of $\mathbb{S}(V^l)$; $\hat{e}_{m_1\ldots m_n}$ is assumed to be properly symmetrised so that the order of the basis labels $m_1\ldots m_n$ is irrelevant. Then using the correspondence $\hat{e}_{m_1\ldots m_n}\leftrightarrow z_{m_1}\ldots z_{m_n}$ and equations (7) and (8), we see that

$$X^l_+\hat{e}_{m_1\ldots m_n}=\sum_m(-1)^m\hat{e}_{m_1\ldots m_n m,-m}$$

$$X^l_-\hat{e}_{m_1\ldots m_n}=\delta_{m_1,-m_2}(-1)^{m_1}\hat{e}_{m_3\ldots m_n}+\text{all possible permutations}$$

while $\hat{n}^l$ is just multiplication by n for any element in the n-particle subspace.

Since $X^l_\pm$, $\hat{n}^l$ all commute with L^l_3, $L^l_\pm$ we actually want the action of the SL(2, $\mathbb{R}$) Lie algebra elements on states $\hat{e}^{L\ n}_{L_3\eta}$, where η is a multiplicity label. The connection between these two bases is given by SO(3) Clebsch-Gordan coefficients:

$$\hat{e}^{L\ n}_{L_3\eta} = \sum_{m_1\dots m_n} \langle m_1 \dots m_n | nLL_3\eta\rangle \hat{e}_{m_1\dots m_n}. \tag{13}$$

Then

$$X^l_\pm \hat{e}^{L\ n}_{L_3\eta} = \sum_{\eta_\pm} K^{\pm\ L\ L_3\ n}_{\ \eta\ \eta_\pm} \hat{e}^{L\ n\pm 2}_{L_3\eta_\pm}.$$

We will not compute the $K^\pm$ coefficients, for our goal is to pass to the limit as $l \to \infty$, in which case the group is no longer SO(3), but E(2), for which the $K^\pm$ coefficients are much easier to compute (see equations (36) and (37)).

Starting with a lowest state (i.e. a state such that X^l_- acting on it annihilates the state), a tower of particles can be generated by repeated application of the raising operator. Such a tower of multiparticle states will carry a (projective) representation of SL(2, $\mathbb{R}$). These towers are labelled by $h > 0$, the irreducible representation label of the positive discrete series of representations of the covering group of SL(2, $\mathbb{R}$) (we use the notation of Sally (1967)), n the number of particles and η the multiplicity parameter. Let $n_{\min}$ be the lowest possible value of n in a given tower; since the spectrum of X^l_0 is given by $2(h+k)$, $k = 0, 1, 2, \dots$, we find from equation (11) that

$$\begin{aligned} 2(h+k) &= \tfrac{1}{2}(2l+1) + n \\ 2h &= l + \tfrac{1}{2} + n_{\min} \end{aligned} \tag{14}$$

which gives the connection between the irreducible representation of SL(2, $\mathbb{R}$) and $n_{\min}$, which is in turn related to L (see equation (2)).

Now choose the scattering operator to be the unitary operator U_g, $g \in \mathrm{SL}(2, \mathbb{R})$ acting on $\mathbb{S}(V^l)$. Then the partial-wave amplitude can be written as

$$\langle nL_3L\eta | S | n'LL_3\eta'\rangle = D^h_{nn'}(g) \tag{15}$$

where g depends on L and η', and η is related to η' through the raising operator that connects the n'-particle subspace to the n-particle subspace. Though the partial-wave amplitude (equation (15)) is unitary, contains 'production' and is invariant with respect to SO(3), it is not the most general partial-wave amplitude satisfying these properties. However equation (15) provides a convenient starting point for constructing more general amplitudes, because it is related to a matrix element of SL(2, $\mathbb{R}$). These matrix elements have been analysed in great detail by Sally.

3. E(2) and SL(2, $\mathbb{R}$)

In § 2 we found amplitudes invariant under SO(3). In this section, by letting $l \to \infty$, we will get amplitudes invariant under E(2). E(2) is the semidirect product of SO(2) and T_2, the translation group in two dimensions. Since E(2) is non-compact, all of its unitary representations are infinite dimensional; they are given as induced representations, induced from one-dimensional representations of T_2. The representation space is $\mathbb{H} = L^2(0, 2\pi)$, and for $f \in \mathbb{H}$, as a function over E(2), it satisfies

$$f(g(\boldsymbol{b}, 0)g(0, \theta)) = \exp(-\mathrm{i}\boldsymbol{p}_0 \cdot \boldsymbol{b}) f(\theta)$$

where $p_0 = \binom{p}{0}$. Group elements are denoted by $g = g(\boldsymbol{b}, \phi) \in \mathrm{E}(2)$, with a composition rule given by

$$g(\boldsymbol{b}, \phi)g(\boldsymbol{b}', \phi') = g(\boldsymbol{b} + R_\phi \boldsymbol{b}', \phi + \phi') \tag{16}$$

with $R_\phi \in \mathrm{SO}(2)$ (we follow the notation of Tung (1985)). The unitary representations $U_{g(\boldsymbol{b},\phi)}$ are given by

$$\begin{aligned}(U_{g(\boldsymbol{b},\phi)}f)(\theta) &= f(g(\mathbf{0}, \theta)g(\boldsymbol{b}, \phi)) \\ &= f(g(R_\theta \boldsymbol{b}, \theta + \phi) \\ &= f(g(R_\theta \boldsymbol{b}, 0)g(\mathbf{0}, \theta + \phi)) \\ &= \exp(-\mathrm{i}\boldsymbol{p}_0 \cdot R_\theta \boldsymbol{b}) f(\theta + \phi).\end{aligned} \tag{17}$$

Momentum wavefunctions can be defined by a map $\mathbb{T}$ to new variables

$$\begin{aligned}\phi(\boldsymbol{p}) &\equiv (\mathbb{T}f)(\boldsymbol{p}) \\ &= f(g(\mathbf{0}, R^{-1}(\boldsymbol{p}))\end{aligned} \tag{18}$$

where $\boldsymbol{p} = R(\boldsymbol{p})\boldsymbol{p}_0$ defines the rotation matrix $R(\boldsymbol{p}) \in \mathrm{SO}(2)$ and

$$(U_{g(\boldsymbol{b},\phi)}\phi)(\boldsymbol{p}) = \exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{b})\phi(R_\phi^{-1}\boldsymbol{p}). \tag{19}$$

On plane wave states this gives

$$U_{g(\boldsymbol{b},\phi)}|\boldsymbol{p}\rangle = \exp(-\mathrm{i}R_\phi \boldsymbol{p} \cdot \boldsymbol{b})|R_\phi \boldsymbol{p}\rangle. \tag{20}$$

A convenient basis in $\mathbb{H} = L^2(0, 2\pi)$ is given by $\hat{e}_m = \mathrm{e}^{\mathrm{i}m\theta}$. The Fock space $\mathbb{S}(\mathbb{H})$ is defined in equation (1), with $\mathbb{H}$ replacing V^l. Basis elements in the n-particle subspace are given by

$$\begin{aligned}\hat{e}_{m_1\ldots m_n} &= \hat{e}_{m_1} \otimes \cdots \otimes \hat{e}_{m_n}|_{\mathrm{sym}} \\ &= \exp\left(\mathrm{i} \sum_{k=-l}^{+l} m_k \theta_k\right) + \text{all permutations.}\end{aligned} \tag{21}$$

$\mathbb{S}(\mathbb{H})$ is isomorphic to a holomorphic Hilbert space HL^2_∞, with the correspondence $\hat{e}_{m_1\ldots m_n} \leftrightarrow z_{m_1} \ldots z_{m_n}$. Let $f(z) \in HL^2_\infty$, where $z = \{z_m\}_{m=-\infty}^{+\infty}$; then the action of E(2) on $f \in HL^2_\infty$ is given by

$$\begin{aligned}(\Gamma_{g(\mathbf{0},\phi)}f)(z) &= f(\mathrm{e}^{-\mathrm{i}m\phi} z_m) \\ (\Gamma_{g(\binom{b}{0},0)}f)(z) &= f\left(\sum_{m'=-\infty}^{+\infty} J_{m-m'}(bp) z_{m'}\right)\end{aligned} \tag{22}$$

where J_m is a Bessel function. For infinitesimal transformations this gives

$$\begin{aligned}(L_3 f)(z) &= \sum_{m=-\infty}^{+\infty} m z_m \frac{\partial}{\partial z_m} f(z) \\ (L_\pm f)(z) &= p \sum_{m=-\infty}^{+\infty} z_{m\pm1} \frac{\partial}{\partial z_m} f(z)\end{aligned} \tag{23}$$

which is a representation of the Lie algebra of E(2) that is irreducible on the one-particle subspace of HL^2_∞, with irreducible representation label p.

We now want to find operators that commute with L_3 and $L_\pm$. In § 2 the operators $\hat{n}^l$ and $X^l_\pm$ were shown to commute with L^l_3 and $L^l_\pm$, the Lie algebra elements of SO(3). In the contraction limit (see Gilmore 1974) SO(3) goes over to E(2). We obtain this contraction limit by defining

$$L_3 = \lim_{l\to\infty} L^l_3$$
$$L_\pm = \lim_{\substack{l\to\infty \\ K\to\infty}} \frac{1}{K} L^l_\pm \tag{24}$$

such that $\lim_{l\to\infty, K\to\infty} l/K = p$. Then

$$[KL_+, KL_-] = 2L_3$$
$$[L_+, L_-] = 2L_3/K^2 \to 0 \qquad \text{as } K\to\infty.$$

Further, from equation (5), $L^l_\pm = \Sigma\, c^\pm_{lm} z_{m\pm1}\, \partial/\partial z_m$ becomes

$$L_\pm = \lim_{\substack{l\to\infty \\ K\to\infty}} \sum_{m=-l}^{+l} \frac{c^\pm_{lm}}{K} z_{m\pm1} \frac{\partial}{\partial z_m}$$
$$= p \sum_{m=-\infty}^{+\infty} z_{m\pm1} \frac{\partial}{\partial z_m} \tag{25}$$

which agrees with equation (23).

Since $X^l_\pm$ commutes with L^l_3 and $L^l_\pm$, we would like to take the limit on $X^l_\pm$ also. However, a problem arises here, in that

$$(X_+f)(z) = \sum_{m=-\infty}^{+\infty} (-1)^m z_m z_{-m} f(z) \tag{26}$$

no longer keeps elements $f \in HL^2_\infty$ in HL^2_∞. On the other hand

$$(X_-f)(z) = \sum_{m=-\infty}^{+\infty} (-1)^m \frac{\partial}{\partial z_m} \frac{\partial}{\partial z_{-m}} f(z) \tag{27}$$

is a perfectly well defined (lowering) operator, and it and $\hat{n}$ commute with L_3, $L_\pm$. Though X^l_+ is the adjoint of X^l_- on HL^2_{2l+1}, X_- on HL^2_∞ no longer has an adjoint.

To see more precisely what this means, we want to compute the action of X_- on n-particle momentum space wavefunctions in $\mathbb{S}(\mathbb{H})$. Using the correspondence between $\hat{e}_{m_1\ldots m_n}$ and $z_{m_1}\ldots z_{m_n}$, we obtain

$$X_-\hat{e}_{m_1\ldots m_n} = (-1)^{m_1}\delta_{m_1,-m_2}\hat{e}_{m_3\ldots m_n} + \text{all permutations} \tag{28}$$

which is similar to equation (13). After some manipulation the action of X_- on n-particle momentum wavefunctions $\phi_n(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$ is given by

$$(X_-\phi_n)(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n-2}) = \int_0^{2\pi} \mathrm{d}\alpha\, \phi_n(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n-2}, \boldsymbol{p}_\alpha, \boldsymbol{p}_{\alpha+\pi}) \tag{29}$$

($\boldsymbol{p}_\alpha = p\binom{\cos\alpha}{\sin\alpha}$), with a corresponding action on n-particle plane-wave states:

$$X_-|\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\rangle_{\text{sym}} = \delta^2(\boldsymbol{p}_1 + \boldsymbol{p}_2)|\boldsymbol{p}_3, \ldots, \boldsymbol{p}_n\rangle_{\text{sym}} + \text{all permutations.} \tag{30}$$

We have seen that X_- does not have an adjoint; the reason for this can now be seen more clearly. We write (formally) $(\phi_n, X_-\phi_{n+2})_{\mathbb{S}(\mathbb{H})} \equiv (X_-^+\phi_n, \phi_{n+2})_{\mathbb{S}(\mathbb{H})}$ and obtain

$$\begin{aligned}(X_+\phi_n)(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n+2}) &\equiv (X_-^+\phi_n)(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n+2}) \\ &= \delta^2(\boldsymbol{p}_1 + \boldsymbol{p}_2)\phi_n(\boldsymbol{p}_3, \ldots, \boldsymbol{p}_{n+2}) + \text{all permutations} \qquad (31)\\ X_+|\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\rangle_{\text{sym}} &= \int_0^{2\pi} \mathrm{d}\alpha |\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, \boldsymbol{p}_\alpha, \boldsymbol{p}_{\alpha+\pi}\rangle_{\text{sym}}.\end{aligned}$$

If $\phi_n \in \mathbb{S}(\mathbb{H})$, $X_+\phi_n \notin \mathbb{S}(\mathbb{H})$ because of the delta functions in equation (31). In that sense X_- does not have an adjoint. However, the action of X_+ on plane-wave states is well defined and shows how n-particle plane waves become $(n+2)$-particle plane waves.

To see how $X_\pm$ generates an SL(2, $\mathbb{R}$) algebra, we want to decompose the n-particle states into direct integrals of irreducible representations of E(2). To obtain this decomposition we use the Mackey double coset machinery (Klink 1969) to reduce n-fold symmetric tensor products. If $f_n(\theta_1, \ldots, \theta_n)$ is an element of the n-particle subspace, set $\theta_i = \theta + D_i$, $i = 1, \ldots, n$, and write

$$(\mathbb{T}_{D_i} f_n)(\theta) = f(\theta + D_1, \ldots, \theta + D_n). \qquad (32)$$

The action of $g(\boldsymbol{b}, \phi) \in \mathrm{E}(2)$ commutes with $\mathbb{T}_{D_i}$ if $\Sigma \sin D_i = 0$. For n-particle momentum space wavefunctions, this gives

$$(\mathbb{T}_{\boldsymbol{k}_i}\phi_n)(\boldsymbol{P}) = \phi_n(R(\boldsymbol{P})\boldsymbol{k}_1, \ldots, R(\boldsymbol{P})\boldsymbol{k}_n) \qquad (33)$$

where

$$\boldsymbol{p}_i = R(\boldsymbol{P})\boldsymbol{k}_i \qquad \sum_{i=1}^{n} \boldsymbol{p}_i = \boldsymbol{P} = R(\boldsymbol{P})\Sigma k_i = R(\boldsymbol{P})\begin{pmatrix} P \\ 0 \end{pmatrix}$$

and P is the irreducible representation label in the direct integral decomposition. The various momenta are given by

$$\boldsymbol{p}_i = p\begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix} \qquad \boldsymbol{k}_i = p\begin{pmatrix} \cos D_i \\ \sin D_i \end{pmatrix}$$

$$\Sigma\boldsymbol{k}_i = p\Sigma\begin{pmatrix} \cos D_i \\ \sin D_i \end{pmatrix} = \begin{pmatrix} p\Sigma\cos D_i \\ 0 \end{pmatrix} = \begin{pmatrix} P \\ 0 \end{pmatrix}.$$

Also

$$\begin{aligned}(U_{g(\boldsymbol{b},0)}\mathbb{T}_{\boldsymbol{k}_i}\phi_n)(\boldsymbol{P}) &= \exp(\mathrm{i}\boldsymbol{b}\cdot\boldsymbol{P})\mathbb{T}_{\boldsymbol{k}_i}\phi_n(\boldsymbol{P}) \\ (U_{g(\boldsymbol{0},\phi)}\mathbb{T}_{\boldsymbol{k}_i}\phi_n)(\boldsymbol{P}) &= (\mathbb{T}_{\boldsymbol{k}_i}\phi_n)(R_\phi^{-1}\boldsymbol{P})\end{aligned} \qquad (34)$$

which agrees with equation (19). The 'double coset' labels $\boldsymbol{k}_i$ do not change under the action of $g(\boldsymbol{b}, \phi)$; they therefore label the multiplicity of representations for a given irreducible representation P.

Now set

$$\hat{\phi}_n(\boldsymbol{P}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n) = (\mathbb{T}_{\boldsymbol{k}_i}\phi_n)(\boldsymbol{P}). \qquad (35)$$

Then

$$\begin{aligned}(X_-\hat{\phi}_n)(\boldsymbol{P}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n-2}) &= \int_0^{2\pi} \mathrm{d}\alpha\, \hat{\phi}_n(\boldsymbol{P}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n-2}, \boldsymbol{k}_\alpha, \boldsymbol{k}_{\alpha+\pi}) \\ X_-|\boldsymbol{P}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n\rangle_{\text{sym}} &= \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2)|\boldsymbol{P}, \boldsymbol{k}_3, \ldots, \boldsymbol{k}_n\rangle_{\text{sym}} + \text{all permutations}\end{aligned} \qquad (36)$$

and

$$(X_+\hat{\phi}_n)(\boldsymbol{P}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n+2})$$
$$= \delta(\boldsymbol{k}_1+\boldsymbol{k}_2)\hat{\phi}_n(\boldsymbol{P}, \boldsymbol{k}_3, \ldots, \boldsymbol{k}_{n+2}) + \text{all permutations} \tag{37}$$
$$X_+|\boldsymbol{P}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n\rangle_{\text{sym}} = \int \mathrm{d}\alpha|\boldsymbol{P}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \boldsymbol{k}_\alpha, \boldsymbol{k}_{\alpha+\pi}\rangle_{\text{sym}}.$$

The states $|\boldsymbol{P}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n\rangle_{\text{sym}}$ are the analogue of $\hat{e}^{L}_{L_3}{}^{n}_{\eta}$ of equation (13), where now η refers to the collection of double coset vectors $\{\boldsymbol{k}_i\}_{i=1}^{n}$.

Finally, we can compute the commutators of $X_\pm$ and $\hat{n}$ on $|\boldsymbol{P}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n\rangle_{\text{sym}}$. $[\hat{n}, X_\pm] = \pm 2X_\pm$ is trivial since $\hat{n}=n$ on an n-particle subspace. Using equations (36) and (37), we obtain

$$[X_-, X_+]|\boldsymbol{P}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n\rangle_{\text{sym}} = (2\pi\delta(0)+2n)|\boldsymbol{P}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n\rangle_{\text{sym}}$$

so that, as an operator relation on plane-wave states 'spanning' $\mathbb{S}(\mathbb{H})$, we have $[X_-, X_+] = I + 2\hat{n}$, which is just the required $\mathrm{SL}(2,\mathbb{R})$ commutation relation. Thus, just as with $\mathrm{SO}(3)\to\mathbb{S}(V^l)$, we have $\mathbb{S}(\mathbb{H})$ carrying a representation of $\mathrm{SL}(2,\mathbb{R})$, invariant under E(2).

The decomposition of the n-particle subspaces of $\mathbb{S}(\mathbb{H})$ differs from that of $\mathbb{S}(V^l)$ given in (2) because the irreducible representations of E(2) are labelled by a continuous positive parameter. On the two-particle subspace P can range from 0 to $2p$, with no multiplicity. However, on a general n-particle subspace ($n>2$), where P ranges between 0 and np, the multiplicity is given by the double coset vectors $\{\boldsymbol{k}_i\}_{i=1}^{n}$. For a fixed $\boldsymbol{P}$, repeated application of $X_\pm$ generates a tower of states that form an irreducible representation of the discrete series of representations of $\mathrm{SL}(2,\mathbb{R})$. Call the unitary operator that acts on this tower of states U_g, $g\in\mathrm{SL}(2,\mathbb{R})$; since U_g commutes with the action of E(2) on $\mathbb{S}(\mathbb{H})$ and is unitary, it can be used as a representation of the scattering operator. As in § 2 the partial-wave amplitudes are then given by

$$\langle n\boldsymbol{P}\{\boldsymbol{k}_i\}|S|n'\boldsymbol{P}\{\boldsymbol{k}'_i\}\rangle = D^{n_{\min}}_{nn'}(g)$$

where $D^{n_{\min}}_{nn'}(g)$ is a matrix element of $\mathrm{SL}(2,\mathbb{R})$ whose irreducible representation is given by the minimum number of particles that can occur in the tower of states generated by repeated application of $X_\pm$ on $|n'\boldsymbol{P}\{\boldsymbol{k}'_i\}\rangle$.

4. Conclusion

We have shown how to construct partial-wave amplitudes that automatically include production reactions, are invariant under the two-dimensional Euclidean group and are unitary. These partial-wave amplitudes are given as matrix elements of the discrete series of representations of $\mathrm{SL}(2,\mathbb{R})$. Such partial-wave amplitudes are not, of course, the most general partial-wave amplitudes that satisfy these properties. For a given n-particle subspace, any operator with a kernel of the form $M^{P,n}_{\{k_i\},\{k'_i\}}$ that mixes the different subenergies also satisfies the properties mentioned above but will not change the particle number.

It is also possible to define raising and lowering operators that connect n to $n\pm N$ particle subspaces and are invariant under the Euclidean group, by writing (for the lowering operator)

$$(X_-^N \phi_n)(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n-N}) = \int \mathrm{d}\alpha_1, \ldots, \mathrm{d}\alpha_N \delta^2(\boldsymbol{q}_1 + \ldots + \boldsymbol{q}_N)\phi_n(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n-N}, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N).$$

These operators are the analogue of operators given in equation (8), where $p(z)$ now has a degree greater than two and is invariant under SO(3). Generally these types of operators do not close to form a finite-dimensional Lie algebra, as was the case for the raising and lowering operators of equations (7), (8), (29) or (31). This can be seen by making use of the fact that the commutator of any two operators commuting with E(2) again gives an operator commuting with E(2) on $\mathbb{S}(\mathbb{H})$. In particular, the commutator of the lowering operator X_-^3 given above with the raising operator X_+ of equation (37) gives an operator that lowers the number of particles by one. By computing repeated commutators of operators that change the number of particles by ± 1, an infinite-dimensional Lie algebra of operators on $\mathbb{S}(\mathbb{H})$ is generated that commutes with E(2). In contrast to SL(2, $\mathbb{R}$) the unitary group action of such an infinite-dimensional Lie algebra on $\mathbb{S}(\mathbb{H})$ is not known.

The argument given here can also be turned around. That is, SL(2, $\mathbb{R}$) operators commute with a much larger group than E(2), namely SO(∞), the set of all unitary operators on $\mathbb{S}(\mathbb{H})$ (for a definition of SO(∞) see Hida (1980)). We have therefore extended Howe's list (Howe 1985) of dual pairs of groups to include infinite parameter groups for both parts of the dual pair.

References

Bargmann V 1962 *Proc. Natl Acad. Sci. USA* **48** 199

Hida T 1980 *Brownian Motion* ed A V Balakrishnan (Berlin: Springer)

Howe R 1985 *Applications of Group Theory in Physics and Mathematical Physics* ed M Flato, P Sally and G Zuckerman (Providence, RI: American Mathematical Society)

Gilmore R 1974 *Lie Groups, Lie Algebras and Some of Their Applications* (New York: Wiley) ch 10

Klink W H 1969 *Lectures in Theoretical Physics* vol XID, ed K T Mahanthappa and W E Brittin (New York: Gordon and Breach) p 43

—— 1987 *J. Phys. A: Math. Gen.* **20** 3565–76

Sally P 1967 *Memoirs of the American Mathematical Society* (Providence, RI: American Mathematical Society)

Segal I 1956 *Trans. Am. Math. Soc.* **81** 106

Tung W K 1985 *Group Theory in Physics* (Singapore: World Scientific) ch 9